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Algebraic Criteria for Entanglement in Multipartite Systems

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Abstract Quantum computing depends heavily on quantum entanglement. It has been known that geometric models for correlated two-state quantum systems (qubits) can be developed using geometric algebra. This suggests that entanglement may be given a purely algebraic description without resort to any particular representation on Hilbert spaces. In the case of the Clifford algebra, for example, the states are not simply operands in a Hilbert space representation of the algebra but they are considered as embedded within the Clifford algebra itself. In other words the space of states sits inside the algebra. This Clifford-algebraic substructure is a minimal left ideal of the algebra. This fact naturally poses the question of whether or not the description of entanglement in multipartite systems can be generalized to algebras possessing one-sided ideal structure. By making tensor products of algebras and their minimal one-sided ideals we propose an algebraic criteria for characterizing entanglement in multipartite systems without resort to any representation on Hilbert spaces.

Keywords Clifford algebra · One-sided ideal · Qubits · Multipartite entanglement

1 Introduction

The definition of entangled states in quantum mechanics is often stated in terms of the factorization properties of tensor products of Hilbert spaces along with the superposition principle. Quantum entanglement in 2-party systems is currently well understood [3, 10, 13]

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but the mathematical description of entanglement for multipartite systems is less understood [10]. Such cases are obviously important not only because their experimental occurrence but also due to their applicability to quantum computation [4, 17]. The main limitation to the understanding of entanglement of such multipartite systems is due to the fact that most of the techniques used to describe entanglement has been developed for 2-party systems. The Schimidt decomposition, for example, [8] which is a procedure for quantifying entanglement, is not valid in general for more than two systems. It turns out that these techniques do not easily generalize to compositions of more than 2-systems. In this work we propose to overcome such difficulties by using methods based on an algebraic approach to entanglement rather than the standard methods based on Hilbert spaces. Such an algebraic approach is then applied to the Clifford algebra. Entangled states will be described by tensor products of algebraic spinors.

Many authors have been investigating the applications of Clifford algebras to describe entanglement and the numerous aspects related to this subject [2, 14, 24]. For many reasons, algebraic and geometric approaches to quantum mechanics and its peculiar features have always been appealing. In this regard, we share with many authors the idea that operators and operands should be elements of the same space. This is in contrast with standard quantum mechanics in which they are treated as elements sitting in separate spaces. As a result one has to work with representations on Hilbert spaces rather than to work with the geometric objects themselves. To our knowledge it was Sauter [21] who first proposed that spinors were elements in a subspace of the Clifford algebra rather than some "column-like" object sitting in a space of representation where elements of the Clifford algebra act as operators. Sauter used this idea to solve Dirac equation using only algebraic techniques. However it was only later that this idea was properly appreciated and clarified in the works of Riesz [18] and Cartan [5]. Despite this, the idea has not been widely exploited in connection with the peculiar nature of quantum phenomena, in particular quantum entanglement. The successful use of Hilbert space representation in quantum mechanics along with its probabilistic interpretation have led to the ideology that Hilbert space structure provides all it is needed to describe quantum physics. However, we believe that it is rather legitimate to investigate richer mathematical structures in connection with quantum theory. This brings additional freedom which might be necessary to improve our understanding of the physical models. For example we may need to rely on algebraic structures in order to allow the introduction of thermodynamics at the quantum level. This has already been exploited in thermofield dynamics [20] and applied to the study of entanglement [19]. A purely algebraic approach to quantum mechanics as opposed to Hilbert space representation was already called for by Dirac [7]. He claimed that such an approach offered a more general way of dealing with situations where it was not possible to represent results in Hilbert space. He gave an explicit example in field theory [6] where description using Hilbert spaces would fail but one could still get results if one used the Heisenberg matrix algebra instead.

In the case of the Clifford algebra of a finite dimensional vector space, a geometric object of great importance is the spinor. The space of spinors within the Clifford algebra describes the quantum kinematics of the theory. These spinor spaces are minimal left ideals [11, 15, 23] of the algebra. They are Clifford modules. One of the importance of this algebraic way of description is that it allows us to see more explicitly that spinors represent an aggregate system [1], and in that sense they represent quantities more naturally associated with statistics and composite systems. Usually these modules can be presented in a Fock-space like form [22, 23]. It then becomes natural the idea of approaching entanglement from tensor products of Clifford algebras, and their minimal one-sided ideals which in turn lead naturally to the description of composition of quantum systems. Furthermore, another advantage of the Clifford algebra with respect to the analysis of entanglement is that the number of particles being analyzed dictates the size of the kinematical space. In this way, results developed in simple cases (such as the 2-party systems) can be more easily generalized to the n-party case.

In this paper we approach entanglement in multipartite systems aiming to apply it to quantum computing. Unlike classical computation, quantum computation includes new possibilities coming from the superposition principle, the so called quantum teleportation for example.

Entangled quantum states provide the means of creating quantum protocols with far reaching abilities to solve tasks with no classical counterpart. Among many examples we refer to algorithms with exponential processing velocities for example [9]. However implementing such formidable applications is still not an easy task. In part most of the problems still relates to a better understanding of the theoretical models for entanglement, particularly the description of entanglement for multipartite systems.

This paper is intended to draw attention to a new line of investigation on entanglement. We state and prove a theorem which establishes the necessary and sufficient conditions for entanglement in a composite system of pure states. Moreover our theorem allows for the identification of non authentic entangled states. Therefore the theorem provides a comprehensive characterization of entanglement in a composite system of pure states.

The paper is unfolded in the following sequence of presentation. We start with algebraic preliminaries recalling the definitions of left ideals, primitive idempotents, and the algebraic quantum space of states. We first show that the tensor product of two elements belonging to two different minimal ideals in an algebra $\mathbb{A}$ results in an element of a minimal left ideal of the tensor product algebra $\mathbb{A} \otimes \mathbb{A}$. Next we generalize this result to any number of systems. In Sect. 3 we state and prove a theorem about tensor product of minimal one-sided ideals which in turn gives the criteria for entanglement. This will be used to describe composite systems. In these initial sections, our approach is general since statements will be made for a general algebra. In Sect. 4 we apply our results to the Clifford Algebra. We use the fact that Dirac bras and kets describing qubits can be naturally associated to elements of minimal one-sided ideals of a Clifford algebra. More precisely we take tensor products of n identical Clifford algebras and use a minimal left ideal of this product algebra as the state space for describing a system of n qubits. Finally we give a description of entanglement in algebraic terms. Section 5 closes with some conclusion remarks. In the appendix the notation and some formal definitions related to spinor space and algebras are presented.

2 Composite Systems in Algebras

In quantum mechanics a composite system is the one that includes more than one quantum object. States for these systems are represented as a tensor product of the individual states of the objects comprising the system [16]. For example, consider a composite system consisting of two subsystems say a and b. Denote by $\mathbf{v} \in H_a$ the state of the system a as an element of the Hilbert space H_a and by $\mathbf{u} \in H_b$ the state of the system b. The state of both systems together is represented as a tensor product $\mathbf{w} = \mathbf{v} \otimes \mathbf{u}$.

Let $\mathbb{A}$ be an algebra over the field $\mathbf{K}$. Every algebra $\mathbb{A}$ which is not nilpotent contains an idempotent element (see Appendix). Moreover, if an algebra $\mathbb{A}$ contains an idempotent, it contains also a primitive idempotent. In algebraic quantum mechanics the algebra $\mathbb{A}$ is then assumed to describe a physical process. The state space of an individual quantum system is represented by the minimal left ideal

$$\mathcal{H} = \mathbb{A}\mathcal{E}, \tag{1}$$

where $\mathcal{E}$ is a primitive idempotent of $\mathbb{A}$. Therefore, the description of composite systems in algebraic quantum mechanics requires tensor products of minimal left ideals. Let us approach such a description starting with simple lemmas and theorems concerning these products.

Lemma 1 *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be primitive idempotents in a simple algebra $\mathbb{A}$. Then $\mathcal{I} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is a primitive idempotent in the algebra $\mathbb{A} \otimes \mathbb{A}$.*

Proof

- Idempotency: $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy $(\mathcal{E}_1)^2 = \mathcal{E}_1$ and $(\mathcal{E}_2)^2 = \mathcal{E}_2$, therefore $\mathcal{I}^2 = (\mathcal{E}_1 \otimes \mathcal{E}_2)(\mathcal{E}_1 \otimes \mathcal{E}_2) = (\mathcal{E}_1\mathcal{E}_1) \otimes (\mathcal{E}_2\mathcal{E}_2) = \mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{I}$.
- Primitivity: An idempotent $\mathcal{J}$ is said to be primitive if $\mathcal{J} A \mathcal{J} = k\mathcal{J}$, for $k \in \mathbf{K}$ and any $A \in \mathbb{A}$. The idempotents $\mathcal{E}_1$ and $\mathcal{E}_2$ are primitive idempotents, i.e. they satisfy $\mathcal{E}_1 A \mathcal{E}_1 = k\mathcal{E}_1$ and $\mathcal{E}_2 A \mathcal{E}_2 = l\mathcal{E}_2$, for $k, l \in \mathbf{K}$ and $A \in \mathbb{A}$. Thus, for $\mathcal{I} = \mathcal{E}_1 \otimes \mathcal{E}_2$ it is easy to check that

$$\begin{aligned}(\mathcal{E}_1 \otimes \mathcal{E}_2)(A \otimes A)(\mathcal{E}_1 \otimes \mathcal{E}_2) &= (\mathcal{E}_1 A \otimes \mathcal{E}_2 A)(\mathcal{E}_1 \otimes \mathcal{E}_2) \\ &= \mathcal{E}_1 A \mathcal{E}_1 \otimes \mathcal{E}_2 A \mathcal{E}_2 \qquad (2) \\ &= c\,\mathcal{E}_1 \otimes \mathcal{E}_2, \qquad (3)\end{aligned}$$

  where $c = kl \in \mathbf{K}$.

□

The results of Lemma 1 can be easily generalized for n-tensor products $\mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A}$.

Lemma 2 *Let $\mathcal{E}_{i_1}, \mathcal{E}_{i_2}, \ldots, \mathcal{E}_{i_n}$ be primitive idempotents in the algebra $\mathbb{A}$. Then the tensor product $\mathcal{E}_{i_1 i_2 \ldots i_n} = \mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2} \otimes \cdots \otimes \mathcal{E}_{i_n}$ is a primitive idempotent in the algebra $(\mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A})_n$ where $(\ )_n$ indicates n factors in the tensor product.*

Next we use these lemmas to prove that the tensor product of n minimal left ideals yields a minimal left ideal.

Theorem 1 *Let $\mathcal{E}_{i_1}, \mathcal{E}_{i_2}, \ldots, \mathcal{E}_{i_n}$ be primitive idempotents in the algebra $\mathbb{A}$ and $A_{r_1}, A_{r_2}, \ldots, A_{r_n}$ arbitrary elements in $\mathbb{A}$. Then the tensor product $A_{r_1}\mathcal{E}_{i_1} \otimes A_{r_2}\mathcal{E}_{i_2} \otimes \cdots \otimes A_{r_n}\mathcal{E}_{i_n}$ is an element of a minimal left ideal in $(\mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A})_n$.*

Proof We simply write,

$$A_{r_1}\mathcal{E}_{i_1} \otimes A_{r_2}\mathcal{E}_{i_2} \otimes \cdots \otimes A_{r_n}\mathcal{E}_{i_n} = (A_{r_1} \otimes A_{r_2} \otimes \cdots \otimes A_{r_n})(\mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2} \otimes \cdots \otimes \mathcal{E}_{i_n})$$

From Lemma 2, we know that $\mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2} \otimes \cdots \otimes \mathcal{E}_{i_n}$ is a primitive idempotent in $(\mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A})_n$ and since $A_{r_1} \otimes A_{r_2} \otimes \cdots \otimes A_{r_n}$ is an arbitrary element of $(\mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A})_n$ it follows that $A_{r_1}\mathcal{E}_{i_1} \otimes A_{r_2}\mathcal{E}_{i_2} \otimes \cdots \otimes A_{r_n}\mathcal{E}_{i_n}$ is an element of minimal left ideal of $(\mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A})_n$ □

3 Algebraic Criteria for Entanglement

In this section we describe an algebraic criteria for entanglement in multipartite systems. Such a criteria will be stated in a form of a theorem. As our interest is entanglement of quantum states related to Pauli spinors, we consider an algebra $\mathbb{A}$ whose generators are $\gamma_1, \gamma_2, \ldots, \gamma_n$ and suppose that $\mathbb{A}$ has one idempotent ε and the basis of the minimal left ideal is given by $(\gamma_i\varepsilon, \gamma_j\varepsilon)$, i, j fixes.

Definition 1 Given a set of generators $\{\gamma_{i_1} \otimes \gamma_{i_2} \otimes \gamma_{i_3} \otimes \cdots \otimes \gamma_{i_n}\}$ in the product algebra $\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A}$, we say that they are in dictionary order [12] in any sum if the elements appear from left to right labeled by a nondecreasing sequence of numbers formed from juxtaposition of their indices for each element.

For example, in the case of the algebra $\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A}$ the generators in the sum, $\gamma_1 \otimes \gamma_1 \otimes \gamma_1 + \gamma_1 \otimes \gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1 \otimes \gamma_3 + \gamma_2 \otimes \gamma_3 \otimes \gamma_1$, are in dictionary order because by considering their indices they appear from left to right with an nondecreasing sequence of numbers $111, 112, 213, 231$.

Theorem 2 *A superposition of basis elements of one sided minimal ideals in the product algebra $\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} \otimes \cdots \otimes \mathbb{A}$ can be decomposed in a tensor product of basis elements of one-sided ideal of each algebra $\mathbb{A}$ if and only if the coefficients ζ_i of the superposition satisfy the relation*

$$\frac{\zeta_1}{\zeta_2} = \frac{\zeta_3}{\zeta_4} = \cdots = \frac{\zeta_{n-1}}{\zeta_n}, \tag{4}$$

where the coefficient ζ_1 is the coefficient of the first element in the dictionary order, ζ_2 is the coefficient of the second and so forth.

Proof Consider any element of the product algebra which is written as the following superposition of basis elements of a minimal left ideal:

$$\begin{aligned}\Psi &= \zeta_1(\gamma_i \otimes \gamma_i \otimes \cdots \otimes \gamma_i)(\varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon)\\ &\quad + \zeta_2(\gamma_i \otimes \gamma_i \otimes \cdots \otimes \gamma_j)(\varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon)\\ &\quad + \cdots\\ &\quad + \zeta_{n-1}(\gamma_j \otimes \gamma_j \otimes \cdots \otimes \gamma_i)(\varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon)\\ &\quad + \zeta_n(\gamma_j \otimes \gamma_j \otimes \cdots \otimes \gamma_j)(\varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon),\end{aligned} \tag{5}$$

where the indices i, j must be such that the product elements appear in dictionary order. If the state Ψ is decomposable (not entangled), it can be written in the form

$$\Psi = (a_1\gamma_i\varepsilon + b_1\gamma_j\varepsilon) \otimes (a_2\gamma_i\varepsilon + b_2\gamma_j\varepsilon) \otimes \cdots \otimes (a_{n-1}\gamma_i\varepsilon + b_{n-1}\gamma_j\varepsilon) \otimes (a_n\gamma_i\varepsilon + b_n\gamma_j\varepsilon). \tag{6}$$

We compute these tensor products to find,

$$\begin{aligned}\Psi &= a_1 a_2 \cdots a_n(\gamma_i \otimes \gamma_i \otimes \cdots \otimes \gamma_i)(\varepsilon \otimes \varepsilon \cdots \otimes \varepsilon)\\ &\quad + a_1 a_2 \cdots b_n(\gamma_i \otimes \gamma_i \otimes \cdots \otimes \gamma_j)(\varepsilon \otimes \varepsilon \cdots \otimes \varepsilon)\\ &\quad + \cdots\end{aligned}$$

$$+b_1b_2\cdots a_n(\gamma_j\otimes\gamma_j\otimes\cdots\otimes\gamma_i)(\varepsilon\otimes\varepsilon\cdots\otimes\varepsilon)$$
$$+b_1b_2\cdots b_n(\gamma_j\otimes\gamma_j\otimes\cdots\otimes\gamma_j)(\varepsilon\otimes\varepsilon\cdots\otimes\varepsilon). \tag{7}$$

By comparing coefficients in the expressions (5) and (7) above we make the following identifications,

$$\begin{aligned}\zeta_1 &= a_1a_2\cdots a_n,\\ \zeta_2 &= a_1a_2\cdots b_n,\\ &\vdots\\ \zeta_{n-1} &= b_1b_2\cdots a_n,\\ \zeta_n &= b_1b_2\cdots b_n.\end{aligned} \tag{8}$$

These relations give rise to a nonlinear algebraic system. We do not need to solve this system in order to determine if the superposition of basis elements of one sided minimal ideals is decomposable. Notice that the system is equivalent to the relation

$$\frac{\zeta_1}{\zeta_2}=\frac{\zeta_3}{\zeta_4}=\cdots=\frac{\zeta_{n-1}}{\zeta_n}=\frac{a_n}{b_n}. \tag{9}$$

It turns out that this expression interrelates all the equations of the nonlinear system. If the system has a solution, then the relation (9) will be satisfied. Conversely, if the system does not have a solution, the relation will not be satisfied. Moreover, recall that the relation is expressed in terms of the coefficients of the state Ψ which was supposed to be decomposable. Therefore a superposition Ψ of basis elements of a one sided minimal ideal will be decomposable if and only if the relation (9) holds. Otherside Ψ is an entangled state. □

The theorem just stated gives a criteria for deciding whether or not a state in a product algebra can be decomposed. The criteria relies basically on (4) which are interrelations of ratios between coefficients in a superposition. These are necessary and sufficient conditions. In consequence we have an algebraic criterion for entanglement of multipartite systems.

4 Application to the Clifford Algebra

In this section we apply our algebraic criteria to the Clifford algebra. We start by discussing how we will approach the qubits in the realm of Clifford algebras. The quickest way of making this connection is via the Pauli algebra. This is because qubits are spinors of the Pauli spin algebra. So we should look at elements of a minimal sided ideal of the Pauli algebra as objects describing the qubits. Let us consider the Pauli algebra $\mathbb{A}$ with its defining relations

$$(\sigma_i)^2=1 \quad (i=1,2,3), \tag{10}$$
$$\sigma_i\sigma_j+\sigma_j\sigma_i=0 \quad (i\neq j). \tag{11}$$

The primitive idempotents are given by:

$$\varepsilon_1=\frac{1}{2}(1+\sigma_3) \quad \text{and} \quad \varepsilon_2=\frac{1}{2}(1-\sigma_3). \tag{12}$$

We can chose one of them, say ε_1 since they can be made equivalent to one another by a similarity transformation. The basis for our left ideal is given by

$$I = A\varepsilon_1 = \{\sigma_1\varepsilon_1, \sigma_3\varepsilon_1\}. \tag{13}$$

Now we can write a qubit as the following element of the minimal left ideal,

$$\Psi = a\sigma_1\varepsilon_1 + b\sigma_3\varepsilon_1, \tag{14}$$

with a and b in **C**. In matrix representation the a and b are identified with the matrix components of spinors Ψ.

Next we write the Bell states,

$$\Gamma := \alpha(\sigma_3\varepsilon_1 \otimes \sigma_3\varepsilon_1) \pm \beta(\sigma_1\varepsilon_1 \otimes \sigma_1\varepsilon_1) = [\alpha(\sigma_3 \otimes \sigma_3) \pm \beta(\sigma_1 \otimes \sigma_1)](\varepsilon_1 \otimes \varepsilon_1), \tag{15}$$

$$\Phi := \eta(\sigma_3\varepsilon_1 \otimes \sigma_3\varepsilon_1) \pm \delta(\sigma_1\varepsilon_1 \otimes \sigma_1\varepsilon_1) = [\eta(\sigma_1 \otimes \sigma_3) \pm \delta(\sigma_3 \otimes \sigma_1)](\varepsilon_1 \otimes \varepsilon_1), \tag{16}$$

with $\alpha, \beta, \eta, \delta \neq 0$.

Notice that these states are elements of the minimal left ideal $(A \otimes A)(\varepsilon_1 \otimes \varepsilon_1)$ in $\mathbb{A} \otimes \mathbb{A}$. We can apply our criteria to verify that these states are indeed entangled. In order to apply the theorem, the first step is to multiply the qubits given in general form (14) and next arrange them in dictionary order for further comparison:

$$\begin{aligned}(a_1\sigma_1\varepsilon_1 + b_1\sigma_3\varepsilon_1) \otimes (a_1\sigma_1\varepsilon_1 + b_1\sigma_3\varepsilon_1) &= \{a_1a_2(\sigma_1 \otimes \sigma_1) + a_1b_2(\sigma_1 \otimes \sigma_3) \\ &\quad + b_1a_2(\sigma_3 \otimes \sigma_1) + b_1b_2(\sigma_3 \otimes \sigma_3)\}(\varepsilon_1 \otimes \varepsilon_1) \\ &= \{\zeta_1(\sigma_1 \otimes \sigma_1) + \zeta_2(\sigma_1 \otimes \sigma_3) + \zeta_3(\sigma_3 \otimes \sigma_1) + \zeta_4(\sigma_3 \otimes \sigma_3)\}(\varepsilon_1 \otimes \varepsilon_1).\end{aligned}$$

We added the last equality in order to make it more explicit the use of our theorem. By the theorem we must have,

$$\frac{\zeta_1}{\zeta_2} = \frac{\zeta_3}{\zeta_4} \quad \Rightarrow \quad \zeta_1\zeta_4 = \zeta_2\zeta_3. \tag{17}$$

Now we compare with expression (15) for the Bell state. We have $\zeta_4 = \alpha \neq 0$, $\zeta_2 = 0$, $\zeta_3 = 0$ and $\zeta_4 = \beta \neq 0$. Hence relation (17) is not satisfied and as expected the state is entangled. Similar analysis also confirms that (16) is entangled.

An interesting example to verify entanglement is the one of two states that differ from one another only by a sign of one of the coefficients. They are the states,

$$\begin{aligned}\Psi_1 &= 1.0(\sigma_1 \otimes \sigma_1)(\varepsilon_1 \otimes \varepsilon_1) + 0.5(\sigma_1 \otimes \sigma_3)(\varepsilon_1 \otimes \varepsilon_1) \\ &\quad + 1.0(\sigma_3 \otimes \sigma_1)(\varepsilon_1 \otimes \varepsilon_1) - 0.5(\sigma_3 \otimes \sigma_3)(\varepsilon_1 \otimes \varepsilon_1),\end{aligned} \tag{18}$$

$$\begin{aligned}\Psi_2 &= 1.0(\sigma_1 \otimes \sigma_1)(\varepsilon_1 \otimes \varepsilon_1) + 0.5(\sigma_1 \otimes \sigma_3)(\varepsilon_1 \otimes \varepsilon_1) \\ &\quad + 1.0(\sigma_3 \otimes \sigma_1)(\varepsilon_1 \otimes \varepsilon_1) + 0.5(\sigma_3 \otimes \sigma_3)(\varepsilon_1 \otimes \varepsilon_1).\end{aligned} \tag{19}$$

By applying our criteria we can easily see that (19) is not entangled whereas (18) is. Let us next consider a three-partite system,

$$\begin{aligned}\Psi &= \alpha(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)(\varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1) + \beta(\sigma_1 \otimes \sigma_1 \otimes \sigma_3)(\varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1) \\ &\quad + \delta(\sigma_1 \otimes \sigma_3 \otimes \sigma_3)(\varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1),\end{aligned} \tag{20}$$

already written in dictionary order. If the state is decomposable it admits the expression,

$$\Psi = (a_1\sigma_1\varepsilon_1 + b_1\sigma_3\varepsilon_1) \otimes (a_2\sigma_1\varepsilon_1 + b_2\sigma_3\varepsilon_1) \otimes (a_3\sigma_1\varepsilon_1 + b_3\sigma_3\varepsilon_1). \tag{21}$$

Computing this tensor product and putting in dictionary order we get,

$$\begin{aligned}\Psi = {} & a_1a_2a_3(\sigma_1\otimes\sigma_1\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) + a_1a_2b_3(\sigma_1\otimes\sigma_1\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) \\ & + a_1b_2a_3(\sigma_1\otimes\sigma_3\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) + a_1b_2b_3(\sigma_1\otimes\sigma_3\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) \\ & + b_1a_2a_3(\sigma_3\otimes\sigma_1\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) + b_1a_2b_3(\sigma_3\otimes\sigma_1\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) \\ & + b_1b_2a_3(\sigma_3\otimes\sigma_3\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\otimes\varepsilon_1) + b_1b_2b_3(\sigma_3\otimes\sigma_3\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1),\end{aligned} \tag{22}$$

which we write as

$$\begin{aligned}\Psi = {} & \zeta_1(\sigma_1\otimes\sigma_1\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) + \zeta_2(\sigma_1\otimes\sigma_1\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) \\ & + \zeta_3(\sigma_1\otimes\sigma_3\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) + \zeta_4(\sigma_1\otimes\sigma_3\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) \\ & + \zeta_5(\sigma_3\otimes\sigma_1\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) + \zeta_6(\sigma_3\otimes\sigma_1\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1) \\ & + \zeta_7(\sigma_3\otimes\sigma_3\otimes\sigma_1)(\varepsilon_1\otimes\varepsilon_1\otimes\otimes\varepsilon_1) + \zeta_8(\sigma_3\otimes\sigma_3\otimes\sigma_3)(\varepsilon_1\otimes\varepsilon_1\otimes\varepsilon_1).\end{aligned} \tag{23}$$

Invoking the theorem, we need to verify the ratios,

$$\frac{\zeta_1}{\zeta_2} = \frac{\zeta_3}{\zeta_4} = \frac{\zeta_5}{\zeta_6} = \frac{\zeta_7}{\zeta_8}. \tag{24}$$

By comparing the states (23) with (20) we find, $\zeta_1 = \alpha$, $\zeta_2 = \beta$ and $\zeta_4 = \delta$ and the remaining ζ_i vanish. We therefore conclude that the state is entangled since $\zeta_1\zeta_4 \neq \zeta_2\zeta_3$.

5 Conclusions and Final Remarks

The aim of this work has been to establish a criteria for entanglement. We have proposed an algebraic approach which we believe it will help to improve the understanding of entanglement in multipartite systems. In this approach, entangled states are described by elements in a minimal left ideal of a tensor product of the algebras which describe the individual systems. A theorem establishing relations among the coefficients of superpositions of elements of the minimal left ideal was given. Those relations provide the necessary and sufficient conditions to decide whether or not a state in a multipartite system is entangled. We have successfully applied the algebraic approach to tensor products of the Pauli algebra confirming expected results.

Concerning the quantification of entanglement, some authors formulate measures of entanglement based on the scalar product of the Hilbert space. We point out that in the algebraic approach an inner product can also be constructed. This is just the canonical pairing between the minimal one-sided ideals. This means that the algebra carries a measure of entanglement and no use of Hilbert space measure is necessary. Among future developments in the algebraic approach to entanglement are: (a) the extension of our results to the statistical operator. This will allow for the description of mixture; (b) the construction of logical gates; and (c) generalization to arbitrary number of dimensions. Works in this direction are in progress and will be presented in forthcoming papers.

Appendix: Algebraic Facts

Recall that a linear algebra $\mathbb{A}=\{A, B, C, \ldots\}$ is a vector space with (i) addition $A+B \in \mathbb{A}$, (ii) multiplication by a scalar $\lambda A \in \mathbb{A}$, λ is an element of the real or complex field, and (iii) a product $AB \in \mathbb{A}$. We will assume that the product to be associative and we consider the algebra to be of finite rank n, with a finite basis $\{e_i\}$.

We assume that algebra $\mathbb{A}$ has a set of idempotents. An idempotent $\mathcal{E}$ is primitive if does not exist two idempotents $\mathcal{E}_1$ and $\mathcal{E}_2$, such that $\mathcal{E}=\mathcal{E}_1+\mathcal{E}_2$. For an primitive idempotent one has $\mathcal{E}A\mathcal{E}=\lambda\mathcal{E}$ where $A\in\mathbb{A}$ and $\lambda \in Z$, the center of $\mathbb{A}$. If $\{\mathcal{E}_i\}$ is a set of primitive idempotents, we can write $\sum_{i=1}^r \mathcal{E}_i$, where $\mathcal{E}_i\mathcal{E}_j=\delta_{ij}\mathcal{E}_j$.

A left ideal, I_L, consists of a set of elements $K\in\mathbb{A}$ such $AK\in I_L, \forall A\in\mathbb{A}$. They are generated by primitive idempotents introduced above. For example, if $\mathcal{E}_i$ is a primitive idempotent, then a minimal left ideal I_L is generated by considering $K=A\mathcal{E}_i$, $\forall A\in\mathbb{A}$ and $K\in I_L$.

Similarly a right ideal I_R is defined through $I_R=\{K\in\mathbb{A}/KA\in I_R, \forall A\in\mathbb{A}\}$ so that a minimal right ideal I_R can be also generated by $\mathcal{E}_i$, giving $K^{'}=\mathcal{E}_iA, \forall A\in\mathbb{A}$ and $K^{'}\in I_R$.

Concluding this appendix we present a result we have obtained about tensor product of identical Clifford algebras: Let V be an n-dimensional linear space over $\mathbf{K}$ endowed with a quadratic form Q, and denote by Cl, a short for $Cl(V,Q)$, the Clifford algebra of V. Then, given Cl with generators γ_i the product algebra $(Cl\otimes Cl\otimes\cdots\otimes Cl)_n$ with generators $\Gamma_{i_1i_2\ldots i_n}:=\gamma_{i_1}\otimes\gamma_{i_2}\otimes\gamma_{i_3}\otimes\cdots\otimes\gamma_{i_n}$ is such that the following relations among the generators hold:

1. $\gamma_i^2=1$,
2. $\gamma_i\gamma_j+\gamma_j\gamma_i=[\gamma_i,\gamma_j]_+=0, i\neq j$,
3. $[\Gamma_{i_1i_2\ldots i_n},\Gamma_{j_1j_2\ldots j_n}]_+=2\delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_n,j_n}1_{(Cl_1\otimes Cl_2\otimes\cdots\otimes Cl_n)}$, if n is odd,
4. $[\Gamma_{i_1i_2\ldots i_n},\Gamma_{j_1j_2\ldots j_n}]_-=0$, if n is even.

In order to verify those relations let us consider:

(a) $[\Gamma_{i_1i_2\ldots i_n}]^2=(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})$,
(b) $[\Gamma_{i_1i_2\ldots i_n},\Gamma_{j_1ji_2\ldots j_n}]_\pm=(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})(\gamma_{j_1}\otimes\gamma_{j_2}\otimes\cdots\otimes\gamma_{j_n})\pm(\gamma_{j_1}\otimes\gamma_{j_2}\otimes\cdots\otimes\gamma_{j_n})(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})$.

Then, for relation (a), we have

$$\begin{aligned}[\Gamma_{i_1i_2\ldots i_n}]^2&=(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})\\&=(\gamma_{i_1}\gamma_{i_1})\otimes(\gamma_{i_2}\gamma_{i_2})\otimes\cdots\otimes(\gamma_{i_n}\gamma_{i_n})=1_{(Cl_1\otimes Cl_2\otimes\cdots\otimes Cl_n)},\end{aligned}$$

where we have used $\gamma_i^2=1$.

For relation (b), if n is odd, we obtain

$$\begin{aligned}&(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})(\gamma_{j_1}\otimes\gamma_{j_2}\otimes\cdots\otimes\gamma_{j_n})\\&=(\gamma_{i_1}\gamma_{j_1})\otimes(\gamma_{i_2}\gamma_{j_2})\otimes\cdots\otimes(\gamma_{i_n}\gamma_{j_n})\\&=(-\gamma_{j_1}\gamma_{i_1})\otimes(-\gamma_{j_2}\gamma_{i_2})\otimes\cdots\otimes(-\gamma_{j_n}\gamma_{i_n})\\&=(-1)^{\text{nodd}}(\gamma_{j_1}\gamma_{i_1})\otimes(\gamma_{j_2}\gamma_{i_2})\otimes\cdots\otimes(\gamma_{j_n}\gamma_{i_n})\\&=-(\gamma_{j_1}\otimes\gamma_{j_2}\otimes\cdots\otimes\gamma_{j_n})(\gamma_{i_1}\otimes\gamma_{i_2}\otimes\cdots\otimes\gamma_{i_n})\end{aligned}$$

or $[\Gamma_{i_1i_2\ldots i_n},\Gamma_{j_1ji_2\ldots j_n}]_+=2\delta_{i_1,j_1}\delta_{i_2,j_{21}}\cdots\delta_{i_n,j_n}1_{(Cl_1\otimes Cl_2\otimes\cdots\otimes Cl_n)}$.

Now, if n is even,

$$[\Gamma_{i_1 i_2 \ldots i_n}, \Gamma_{j_1 j i_2 \ldots j_n}]_- = 0.$$

These results show that for a number n of systems we have a different product algebra structure if n is even or odd.

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